

# Function

A *function of one independent variable* is a relation in the form  $y = f(x)$  such that there exists one and only one value of  $y$  in the range of  $f$  for each real number  $x$  in the domain of  $f$ . The variable  $y$  is called the *dependent variable*.

An *implicit function* is a function in which both dependent- and independent variables appear on the same side.

An *explicit function* is one where the dependent variable is on the left hand side-, and the independent variable on the right hand side of the equation.

# Polynomial

$$a_n x^n + \dots + a_1 x + a_0$$

## Polynomial equation

$$a_n x^n + \dots + a_1 x + a_0 = 0$$

## Quadratic equation

$$ax^2 + bx + c = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

## Definition of limits

If  $f(x)$  is a function which draws closer to a unique finite real number  $l$  for all values of  $x$  as the latter draws closer to  $a$ , but  $x \neq a$ , then  $l$  is called the *limit* of  $f(x)$  as  $x$  approaches  $a$ . In notation this is,

$$\lim_{x \rightarrow a} f(x) = l$$

## Definition of limits

For a function  $f(x)$ ,  $\lim_{x \rightarrow a} f(x) = l$  if and only if for every  $\epsilon > 0$ , there exists  $\delta > 0$  such that  $|f(x) - l| < \epsilon$  whenever  $0 < |x - a| < \delta$ .

# Rules of limits

$$\lim_{x \rightarrow a} k = k$$

$$\lim_{x \rightarrow a} x^n = a^n$$

$$\lim_{x \rightarrow a} kf(x) = k \lim_{x \rightarrow a} f(x)$$

$$\lim_{x \rightarrow a} [f(x) \pm g(x)] = \lim_{x \rightarrow a} f(x) \pm \lim_{x \rightarrow a} g(x)$$

$$\lim_{x \rightarrow a} [f(x) \cdot g(x)] = \lim_{x \rightarrow a} f(x) \cdot \lim_{x \rightarrow a} g(x)$$
$$\lim_{x \rightarrow a} \left[ \frac{f(x)}{g(x)} \right] = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)}, \text{ provided that } \lim_{x \rightarrow a} g(x) \neq 0$$
$$\lim_{x \rightarrow a} [f(x)]^n = \left[ \lim_{x \rightarrow a} f(x) \right]^n, \text{ provided that } n > 0$$

## Definition of a derivative

Let  $y = f(x)$ . Then, the derivative of  $y$  with respect to  $x$  is,

$$\frac{dy}{dx} = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

The various notations for the derivative include

$$\frac{df(x)}{dx}, \frac{df}{dx}, f'(x), y', Dy \text{ and } D(f(x))$$

# General rules of differentiation

Let  $u$ ,  $v$  and  $w$  are functions of  $x$ , and  $c$  is a constant.

$$\frac{d}{dx}(c) = 0$$

$$\frac{d}{dx}(cx) = c$$

$$\frac{d}{dx}(cx^n) = ncx^{n-1}$$

$$\frac{d}{dx}(u \pm v \pm \dots) = \frac{du}{dx} \pm \frac{dv}{dx} \pm \dots$$

$$\frac{d}{dx}(cu) = c \frac{du}{dx}$$

$$\frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx}$$

$$\frac{d}{dx}(uvw) = uv \frac{dw}{dx} + uw \frac{dv}{dx} + vw \frac{du}{dx}$$

$$\frac{d}{dx} \left( \frac{u}{v} \right) = \frac{v \left( \frac{du}{dx} \right) - u \left( \frac{dv}{dx} \right)}{v^2}$$

$$\frac{d}{dx}(u^n) = nu^{n-1} \frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$$

$$\frac{du}{dx} = \frac{1}{\frac{dx}{du}}$$

$$\frac{dy}{dx} = \frac{\frac{dy}{du}}{\frac{dx}{du}}$$

$$\frac{de^x}{dx} = e^x$$

$$\frac{d \ln x}{dx} = \frac{1}{x}$$

# Laws of exponents

Let  $p$  and  $q$  be real numbers,  $a$  and  $b$  positive numbers, and  $m$  and  $n$  positive integers. Then,

$$a^p \cdot a^q = a^{p+q}$$

$$\frac{a^p}{a^q} = a^{p-q}$$

$$(a^p)^q = a^{pq}$$

$$a^0 = 1, \text{ provided that } a \neq 0$$

$$\begin{aligned}a^{-p} &= \frac{1}{a^p} \\(ab)^p &= a^p b^p \\\sqrt[n]{a} &= a^{\frac{1}{n}} \\\sqrt[n]{a}^m &= a^{\frac{m}{n}} \\\sqrt[n]{\frac{a}{b}} &= \frac{\sqrt[n]{a}}{\sqrt[n]{b}}\end{aligned}$$

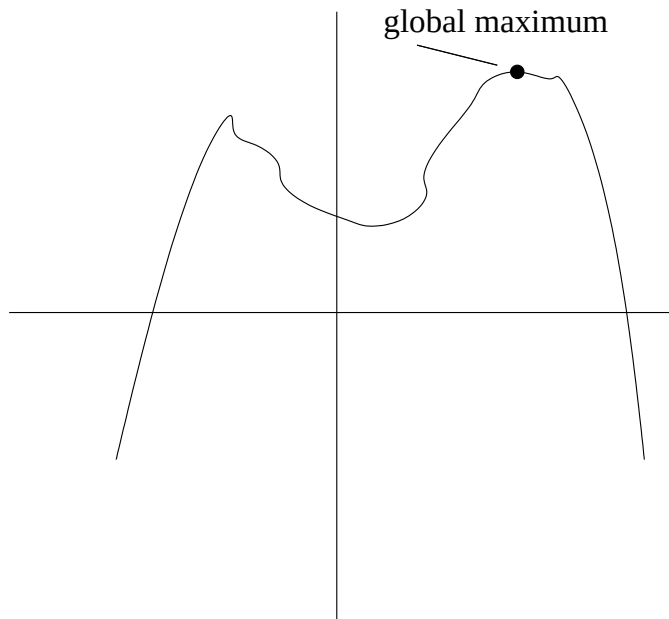
# Laws of logarithms

$$\log_a mn = \log_a m + \log_a n$$

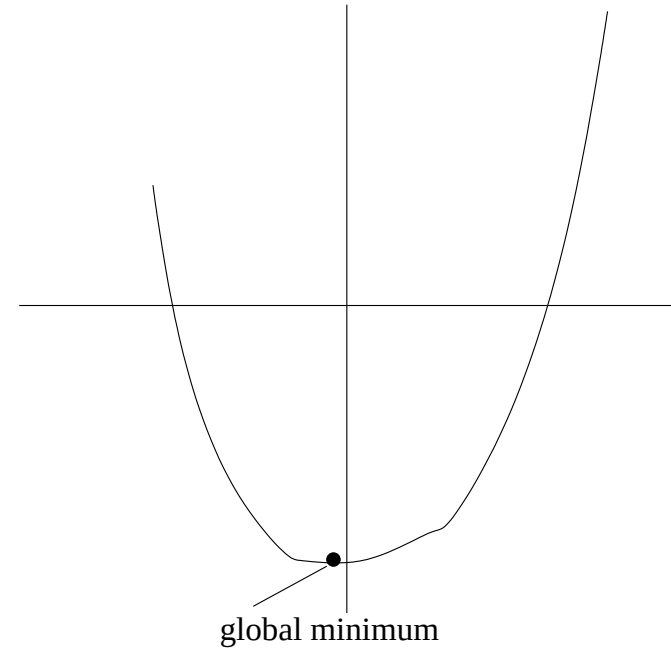
$$\log_a \frac{m}{n} = \log_a m - \log_a n$$

$$\log_a m^p = p \log_a m$$

$$\log_a n = \frac{\log_b n}{\log_b a}$$



(a)



(b)

Figure 1. (a) *global maximum*-, and (b) *global minimum* points of a function.

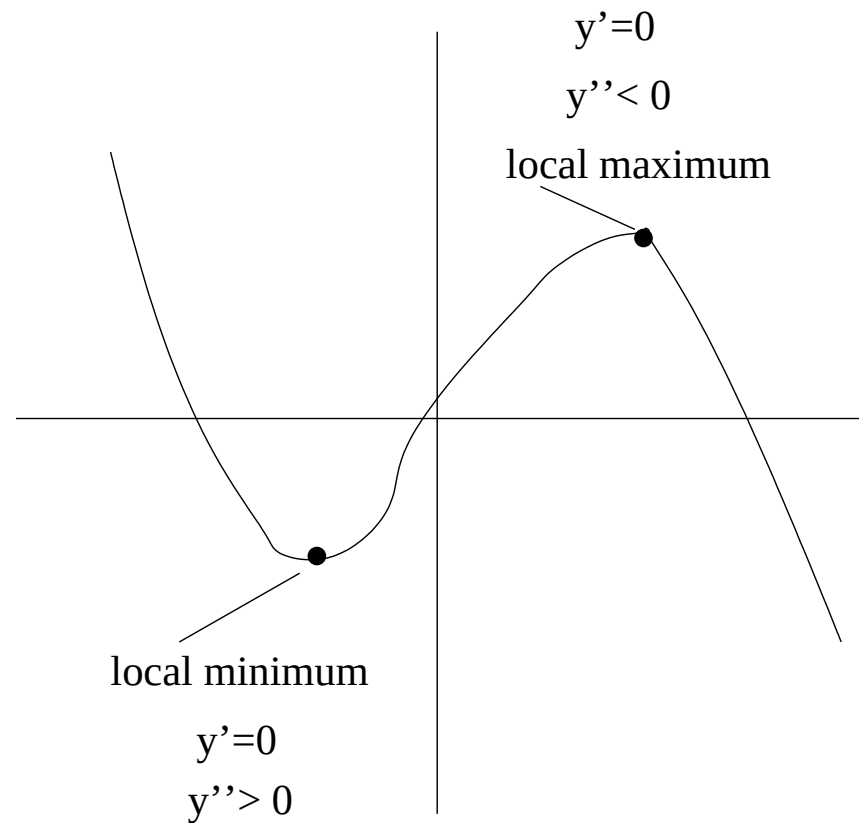


Figure 2. *Local minimum- and local maximum points of a function.*

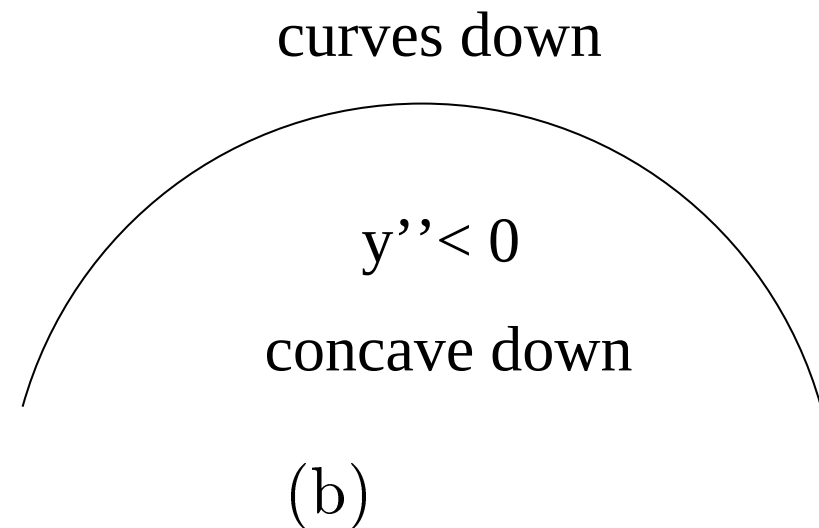
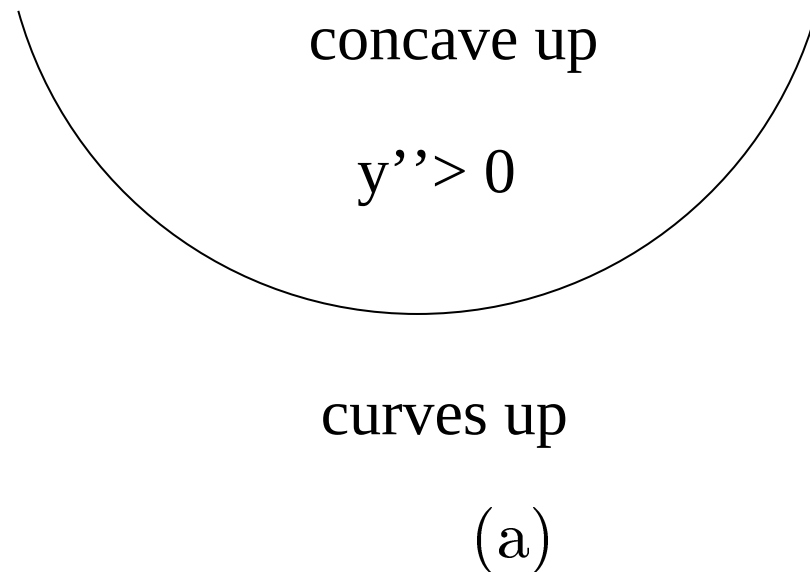


Figure 3. *The two curvature types, namely (a) concave up ( $y'' > 0$ ) and, (b) concave down ( $y'' < 0$ ).*

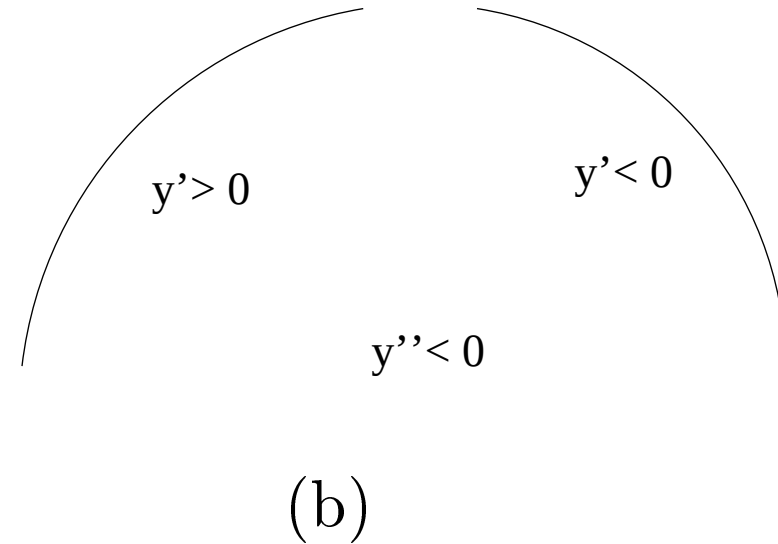
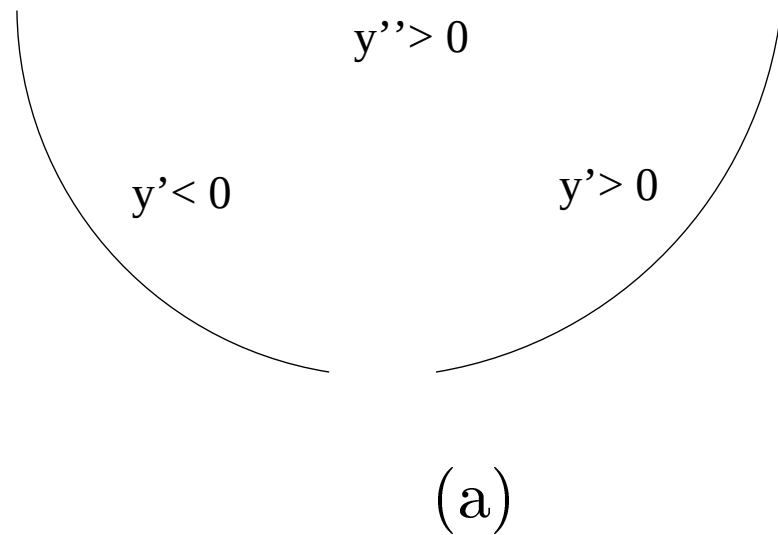


Figure 4. *The four possible curvatures in two dimensions, considering both  $y'$  and  $y''$ , (a)  $y'' > 0$  and, (b)  $y'' < 0$ .*

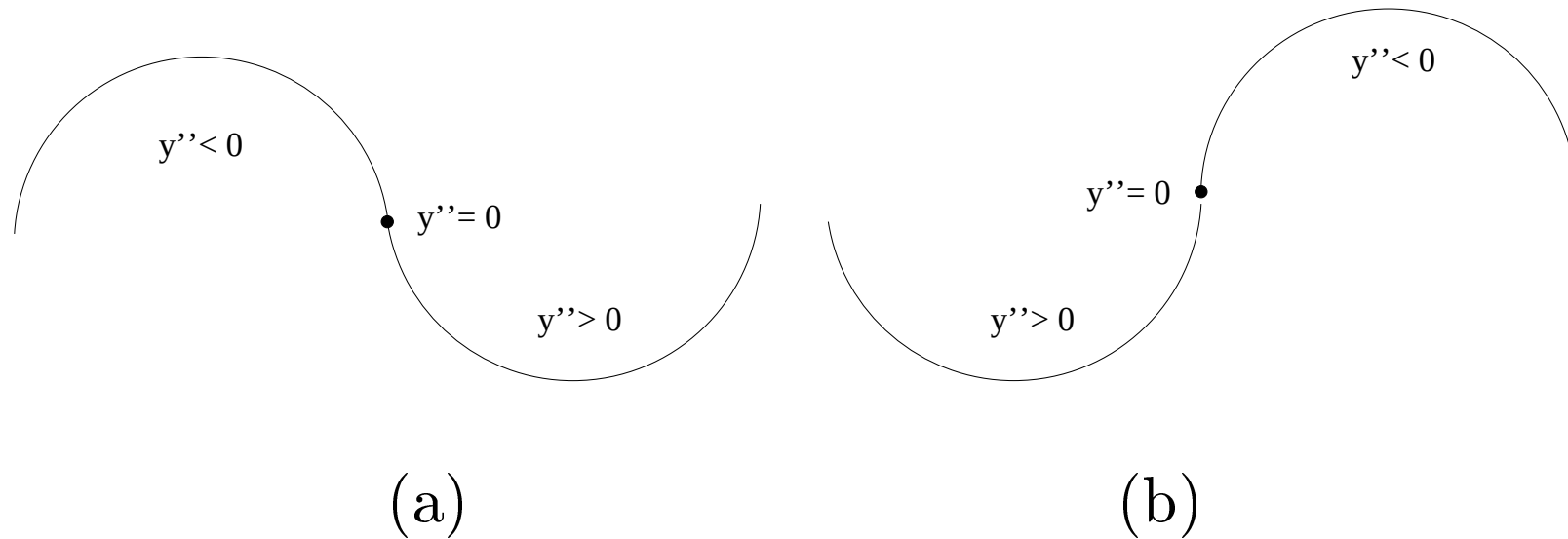


Figure 5. *Inflection points, where  $y'' = 0$ , (a) with  $y''$  increasing and, (b) with  $y''$  decreasing.*

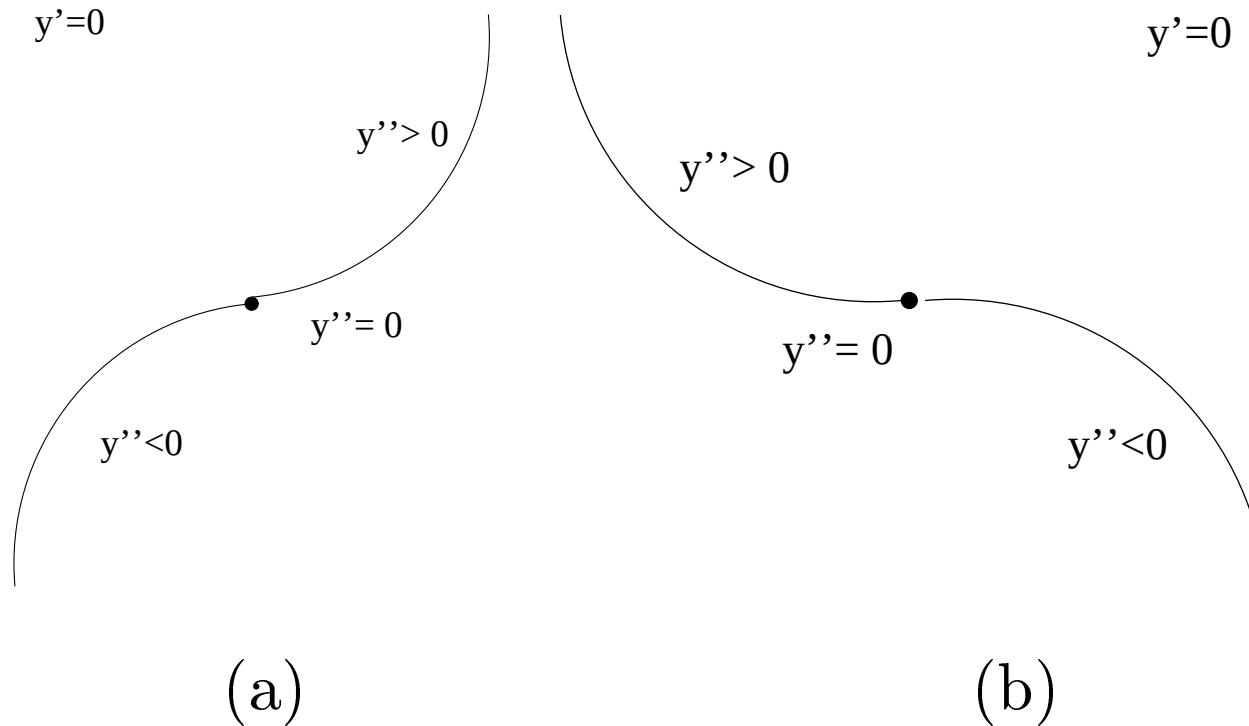


Figure 6. *Stationary inflection points, where both  $y' = 0$  and  $y'' = 0$ , (a) with  $y''$  increasing and, (b) with  $y''$  decreasing.*

Figure 7 shows a plot of the case where  $a = 1$ ,  $b = 2$ ,  $c = 3$  and  $d = 4$ .

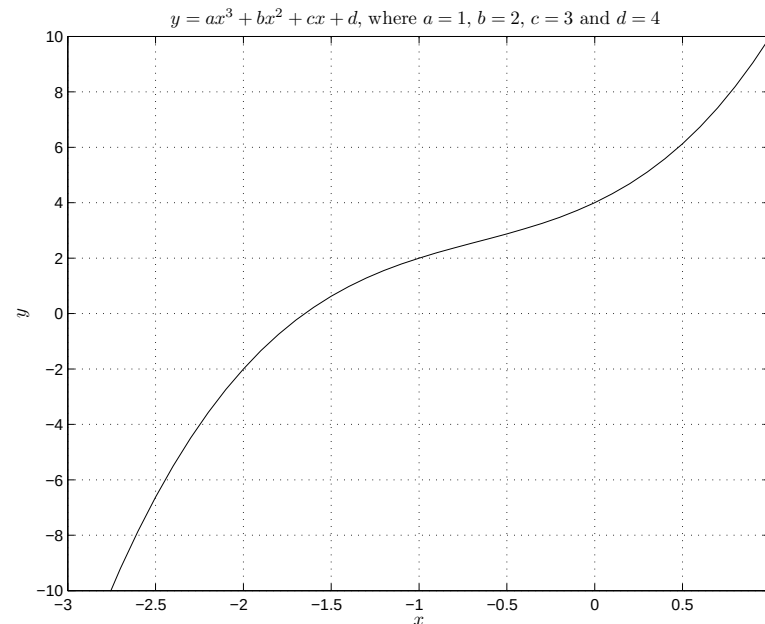


Figure 7. *The cubic function  $y = x^3 + 2x^2 + 3x + 4$ .*

Figure 8 shows the case where  $a = 1$ ,  $b = 5$ ,  $c = 4$  and  $d = 3$

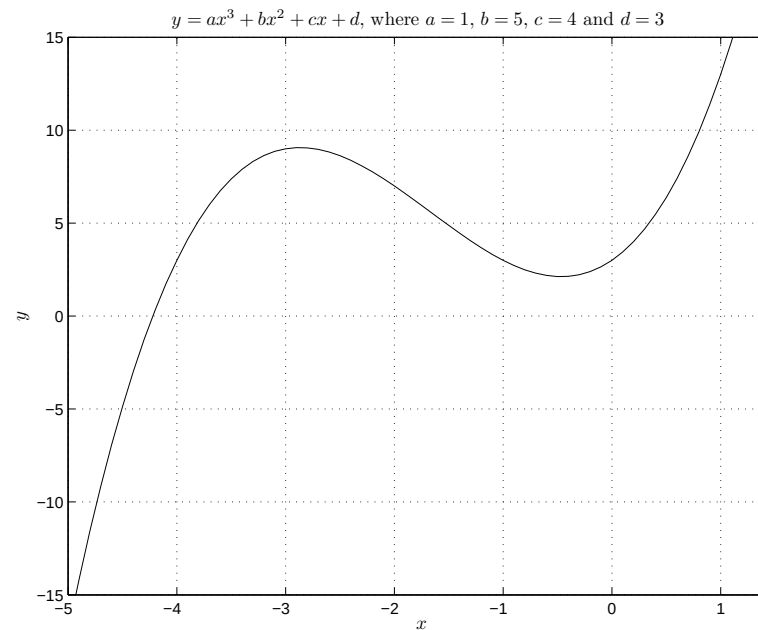


Figure 8. *The cubic function  $y = x^3 + 5x^2 + 4x + 3$ .*

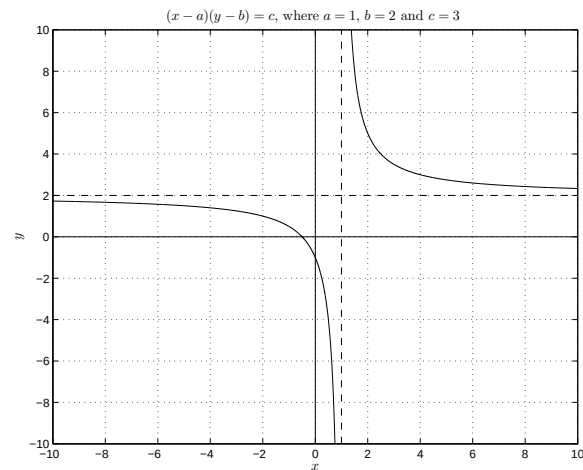


Figure 9. *The hyperbolic function  $(x - 1)(y - 2) = 3$ .*

The *general demand function* is of the form

$$q_d = f(p, y, p_s, p_c, t_a, a, \dots)$$

where  $q_d$  is the quantity demand of good  $x$ ,  $p$  the price of  $x$ ,  $y$  the income of the consumer,  $p_s$  the price of substitute goods,  $p_c$  the price of complementary goods,  $t_a$  the taste or fashion of the consumer, and  $a$  the advertisement level.

In its simplest case where all other factors are constant, the *demand equation* takes the form

$$p = c_1 - c_2 q_d$$

where  $p$  is the price-, while  $q_d$  the quantity demanded of the good  $x$ , and  $c_1$  and  $c_2$  are positive constants.

The *general supply function* is of the form

$$q_s = f(p, c, p_0, t_e, n, o, \dots)$$

where  $q_s$  is the quantity supplied of good  $x$ ,  $p$  the price of  $x$ ,  $c$  the cost of production,  $p_0$  the price of other goods,  $t_e$  the available technology,  $n$  the number of producers in the market, and  $o$  other factors, for example tax and subsidies.

The simplified relation for the *supply* is

$$p = c_1 + c_2 q_s$$

where  $q_s$  is the quantity of  $x$  supplied, and  $c_1$  and  $c_2$  are positive constants.

Definition 1. The *total cost*  $c_t$  comprises a fixed cost and a variable cost, that is

$$c_t = c_f + c_v$$

The *total revenue*  $r_t$  is price times output,

$$r_t = pq$$

**Definition 2.** The *marginal cost*  $c_m$  is the change in total cost caused by the production of an additional unit.

The *marginal revenue*  $r_m$  is the change in total revenue coming from the sale of an extra good. In other words,

$$c_m = \frac{dc_t}{dq} \text{ and}$$

$$r_m = \frac{dr_t}{dq}$$

where  $q$  is the output.

**Definition 3.** The *average cost*  $c_a$  is the total cost per unit output,

$$c_a = \frac{c_t}{q}$$

The *average revenue*  $r_a$  is the total revenue per unit output,

$$r_a = \frac{r_t}{q}$$

From Definition 1,  $r_t = pq$ , and from Definition 3,  $r_t = r_a q$ , therefore

$$p = r_a$$

From Definition 3,  $c_a = \frac{c_t}{q}$ , and from Definition 1,  $c_a = c_f + c_v$ . Therefore

$$c_a = c_{af} + c_{av}$$

where the average fixed cost  $c_{af} = \frac{c_f}{q}$  and the average variable cost  $c_{av} = \frac{c_v}{q}$ .

Procedure 1. *Total-, average-, and marginal graphs*

Given:  $c_t(x)$

$$c_a \leftarrow \frac{c_t}{x}$$

$$c_m \leftarrow c'_t$$

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for each  $f \in \{c_t, c_a, c_m\}$  do  
  find the critical values  $\mathbf{x}_c$  for  $f' = 0$   
  find  $f''$   
   $n \leftarrow |\mathbf{x}_c|$   
  for  $i = 1$  to  $n$  do  
    if  $f''(x_i^c) > 0$  then  
       $f(x_i^c)$  is convex and is the relative minimum of  $f$   
    elseif  $f''(x_i^c) < 0$  then  
       $f(x_i^c)$  is concave and is the relative maximum of  $f$   
    endif  
  endfor  
  find inflection points  $\mathbf{x}_f$  from  $f'' = 0$   
endfor  
plot  $c_t(x)$ , then  $c_a(x)$  and  $c_m(x)$ 
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The relationship between input and output is called a *production function*,

$$q = f(l, k, r, t_e, s, e, \dots)$$

where  $l$  is labour,  $k$  physical capital such as buildings and machines,  $r$  raw materials,  $t_e$  technology,  $s$  land, and  $e$  enterprise.

Assuming a short period of time, then  $l$  becomes the only independent variable and the other remaining factors are parameters, that is fixed, and therefore  $q = f(l)$ .

Then the *marginal product of labour* is

$$p_{lm} = \frac{dq}{dl}$$

and the *average product of labour* is

$$p_{la} = \frac{q}{l}$$

The *marginal propensity to consume* is  $p_{cm} = \frac{dc}{dy}$ .

The *marginal propensity to save* is  $p_{sm} = \frac{ds}{dy}$ .

The *average propensity to consume* is  $p_{ca} = \frac{c}{y}$ .

The *average propensity to save* is  $p_{sa} = \frac{s}{y}$ .

Here  $y$  is the income,  $c$  the consumption, and  $s$  the saving.

The *profit* is

$$\pi = r_t - c_t$$

At the *break-even point*

$$\pi = 0$$

that is

$$r_t = c_t$$